18.01A Recitation — Monday, Sept. 10, 2018

Quick Review:

- Given $y = f(x)$.
  - Linear approximation at $x = a$: $f(x) \approx f(a) + f'(a)(x-a)$.
  - Quadratic approximation at $x = a$: $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$.
- Useful approximation formula at $x = 0$:
  1. $e^x \approx 1 + x + \frac{x^2}{2}$
  2. $\sin x \approx x$
  3. $\cos x \approx 1 - \frac{x^2}{2}$
  4. $\frac{1}{1-x} \approx 1 + x + x^2$
  5. $(1+x)^a \approx 1 + ax + \frac{a(a-1)}{2}x^2$
  6. $\ln(1+x) \approx x - \frac{1}{2}x^2$
- L'Hôpital's rule: if when $x \to a$, $\frac{f(x)}{g(x)}$ is either $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

  provided that $f'(x)$ and $g'(x)$ are defined near a and the right side limit exists. L'Hôpital's rule can be applied repeatedly if the right side is again either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Practice problems:

1. Find the quadratic approximation of $\cos(5x)$ at $x = 0$
   (a) by using the general formula $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$.
   (b) by using the quadratic approximation $\cos x \approx 1 - \frac{x^2}{2}$.

   Compare the two results.

2. Find the quadratic approximation of

$$\frac{1}{(1-2x)(1-3x)}$$

   at $x = 0$ by using the basic approximation formulas.

3. Find the quadratic approximation of

$$\frac{(1+x)^{\frac{3}{2}}}{1+2x}$$

at $x = 0$ by using the basic formulas.

4. Evaluate the following limits.

   (a)
   $$\lim_{x \to -\infty} xe^x$$

   (b)
   $$\lim_{x \to 0} x^{x^2}$$

   (c)
   $$\lim_{x \to 0} \frac{\sin 2x - 2\sin x}{\sin 3x - 3\sin x}$$

   (d)
   $$\lim_{x \to \frac{\pi}{4}} \frac{\ln(\tan x)}{\sin x - \cos x}$$

   (e)
   $$\lim_{x \to 0} \frac{e^{2x} - 1}{\sin 5x}$$

   (f)
   $$\lim_{x \to \infty} \frac{\ln(\ln x)}{\ln x}$$

   (g)
   $$\lim_{x \to 0} (\cos x)^{\frac{1}{x}}$$

   (h)
   $$\lim_{x \to \infty} e^{-x} \ln x$$

   (i) (This is a hard one.)
   $$\lim_{x \to 0} \cot x - \frac{1}{x}$$